

# On the $L^p$ -distortion of finite quotients of amenable groups.

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## Abstract

We study the  $L^p$ -distortion of finite quotients of amenable groups. In particular, for every  $2 \leq p < \infty$ , we prove that the  $\ell^p$ -distortions of the groups  $C_2 \wr C_n$  and  $C_{p^n} \rtimes C_n$  are in  $\Theta((\log n)^{1/p})$ , and that the  $\ell^p$ -distortion of  $C_n^2 \rtimes_A \mathbf{Z}$ , where  $A$  is the matrix  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$  is in  $\Theta((\log \log n)^{1/p})$ .

## 1 The main results

### 1.1 Distortion

Let us first recall some basic definitions.

**Definition 1.1.**

- Let  $0 < R \leq \infty$ . The distortion at scale  $\leq R$  of an injection between two discrete metric spaces  $F : (X, d) \rightarrow (Z, d)$  is the number (possibly infinite)

$$dist_R(F) = \sup_{0 < d(x,y) \leq R} \frac{d(f(x), f(y))}{d(x, y)} \cdot \sup_{0 < d(x,y) \leq R} \frac{d(x, y)}{d(f(x), f(y))}.$$

If  $R = \infty$ , we just denote  $dist(F)$  and call it the distortion of  $F$ .

- The  $\ell^p$ -distortion  $c_p(X)$  of a finite metric space  $X$  is the infimum of all  $dist_F$  over all possible injections  $F$  from  $X$  to  $\ell^p$ .

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Let  $G$  be a finitely generated group. Let  $S$  be a symmetric finite generating subset of  $G$ . We equip  $G$  with the left-invariant word metric associated to  $S$ :  $d_S(g, h) = |g^{-1}h|_S = \min\{n \in \mathbf{N}, g^{-1}h \in S^n\}$ . Let  $(G, S)$  denote the associated Cayley graph of  $G$ : the set of vertices is  $G$  and two vertices  $g$  and  $h$  are joined by an edge if there is  $s \in S$  such that  $g = hs$ . Note that the graph metric on the set of vertices on  $(G, S)$  coincides with the word metric  $d_S$ .

Let  $\lambda_{G,p}$  denote the regular representation of  $G$  on  $\ell^p(G)$  for every  $1 \leq p \leq \infty$  (i.e.  $\lambda(g)f(x) = f(g^{-1}x)$ ). The  $\ell^p$ -direct sum of  $n$  copies of  $\lambda_{G,p}$  will be denoted by  $n\lambda_{G,p}$ .

Our main results are the following theorems.

**Theorem 1.** *Let  $m$  be an integer  $\geq 2$ . For all  $n \in \mathbf{N}$ , consider the finite lamplighter group  $C_m \wr C_n = (C_m)^{C_n} \rtimes C_n$  equipped with the generating set  $S = ((\pm 1_0, 0), (0, \pm 1))$ , where  $1_0 \in (C_m)^{C_n}$  is the characteristic function of the singleton  $\{0\}$ . For every  $2 \leq p < \infty$ , there exists  $C = C(p, m) < \infty$  such that*

$$C^{-1}(\log n)^{1/p} \leq c_p(C_m \wr C_n, S) \leq C(\log n)^{1/p}.$$

Note that the upper bound has been very recently proved for  $p = 2$  by Austin, Naor, and Valette [ANV], using representation theory. The proof that we propose here is shorter and completely elementary. On the other hand, the lower bound was known (see [LNP], or Section 2).

**Theorem 2.** *Let  $m$  be an integer  $\geq 2$ . For all  $n \in \mathbf{N}$ , consider the group  $BS_{m,n} = C_m^n \rtimes C_n$  equipped with the generating set  $S = \{(\pm 1, 0), (0, \pm 1)\}$ . For every  $2 \leq p < \infty$ , there exists  $C = C(p, m) < \infty$  such that*

$$C^{-1}(\log n)^{1/p} \leq c_p(G_n, S) \leq C(\log n)^{1/p}.$$

**Theorem 3.** *For all  $n \in \mathbf{N}$ , consider the group  $SOL_n = C_n \rtimes_A C_{o(A,n)}$ , where  $A$  is a matrix of  $SL_2(\mathbf{Z})$  with eigenvalues of modulus different from 1, e.g. the matrix  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ , and where  $o(A, n)$  denotes the order of  $A$  in  $SL_2(C_n)$ . Equip  $G$  with the generating set  $S = \{(\pm 1, 0), (0, \pm 1)\}$ . For every  $2 \leq p < \infty$ , there exists  $C = C(p) < \infty$  such that*

$$C^{-1}(\log \log n)^{1/p} \leq c_p(G_n, S) \leq C(\log \log n)^{1/p}.$$

## 1.2 About the constructions

We will say that map  $F : G \rightarrow E$  from a group  $G$  to a Banach space is equivariant if it is the orbit of 0 of an isometric affine action of  $G$  on  $E$ . Let  $\sigma$  be such

an action. The equivariance of  $F(g) = \sigma(g).0$  implies that  $\|F(g) - F(h)\| = \|F(g^{-1}h)\|$ . Hence the distortion at scale  $\leq R$  of  $F$  is just given by

$$\text{dist}_R(F) = \sup_{0 < |g|_S \leq R} \frac{|g|_S}{\|F(g)\|} \cdot \sup_{0 < |g|_S \leq R} \frac{\|F(g)\|}{|g|_S}.$$

All the groups involved in the main theorems are of the form  $G = N \rtimes A$  where  $A$  is a finite cyclic group. To prove an upper bound on  $c_p(G)$ , our general approach is to construct an embedding  $F = F_1 \oplus^{\ell^p} F_2$ , where  $F_1$  is the orbit of 0 of an affine action  $\sigma_1$  of  $G$ , whose linear part is  $K\lambda_{G,p}$  (for some  $K \in \mathbf{N}$ ), and such that for  $R = \text{Diam}(N)$ , we have

$$\text{dist}_R(F_1) \approx (\log R)^{1/p}.$$

More precisely, for  $F_{m,n}$  and  $BS_{m,n}$  (resp. for  $SOL_{A,n}$ ), we will need  $K \approx \log(mn)$  (resp.  $K \approx \log \log n$ ) copies of  $\lambda_{G,p}$ .

For  $G = F_{m,n}$  or  $BS_{m,n}$ , we can take  $F = F_1$  since  $\text{Diam}(N) \approx \text{Diam}(G) \approx n$  (see Proposition 3.1). But, for  $G = SOL_{A,n}$ , we have  $\text{Diam}(N) \approx \log n$ , which can be much less than  $\text{Diam}(G) \approx o(A, n)$ . Hence, the solution in this case is to add some map  $F_2 : G/N \approx C_{o(A,n)} \rightarrow \ell^p$  with a bounded distortion (for instance, take the orbit of 0 under the action of  $C_{o(A,n)}$  on  $\mathbf{R}^2$  such that 1 acts by rotation of center  $(o(A, n), 0)$  and angle  $2\pi/o(A, n)$ ).

Note that Theorem 3 also holds for the group  $C_n \rtimes_A \mathbf{Z}$ , in which case we can take an action of  $\mathbf{Z}$  by translations on  $\mathbf{R}$  to embed the quotient with bounded distortion (i.e. for  $F_2$ ).

## 2 Upper bounds on the distortion

Let  $1 \leq p \leq \infty$ . Recall [T1] that the left- $\ell^p$ -isoperimetric profile in balls of  $(G, S)$  is defined by

$$J_{G,S,p}(n) = \sup_{\text{Supp}(f) \subset B(1,n)} \frac{\|f\|_p}{\sup_{s \in S} \|\lambda(s)f - f\|_p},$$

where  $B(1, n)$  denotes the open ball of radius  $n$  and center 1 in  $(G, S)$ . For convenience, we will

Our main result in [T1] consisted in showing that a lower bound on the isoperimetric profile can be used to construct metrically proper affine isometric actions of  $G$  on  $\ell^p(G)$  whose compressions satisfy lower bounds which are optimal in certain cases. Here, we will use it to produce upper bounds on the  $\ell^p$ -distortion of finite groups.

On the other hand, as explained in [T2], if  $X = (G, d_S)$  is a Cayley graph, then the inequality  $J_{p,G} \geq J$  for some non-decreasing function  $J : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  implies Property A(J,p) (see [T2, Definition 4.1]) for the space  $X$  (if the group  $G$  is amenable, a standard average argument actually shows that this is an equivalence). So in a large extend, the results of the present paper are easy consequences of the method explained in [T2].

A crucial remark is that  $J_{G,S,p}$  is a local quantity, and hence behaves well under quotients. Namely, we recall the following easy fact.

**Proposition 2.1.** *(for a proof, see [T3, Theorem 4.2]) Let  $\pi : G \rightarrow Q$  be a surjective homomorphism between two finitely generated groups and let  $S$  be a symmetric generating subset of  $G$ . Then*

$$J_{G,S,p} \leq J_{Q,\pi(S),p}.$$

Our main technical tool is the following proposition, which is an analogue of [T2, Proposition 4.5]. For the convenience of the reader, we give its relatively short proof in Section 4.

**Theorem 4.** *Let  $X = (G, S)$  be a finite Cayley graph such that  $J_{G,S,p}(r) \geq J(r)$  when  $r \leq R$ , for some  $R \leq \text{Diam}(G)/2$ . Then, there exists an affine isometric action  $\sigma$  of  $G$  on such that*

- *the linear part of  $\sigma$  is the  $\ell^p$ -direct sum of  $K = \lfloor \log R \rfloor$  regular representations of  $G$  in  $\ell^p(G)$ .*
- *The orbit of 0 induces an injection  $F : G \rightarrow \bigoplus_{k=0}^{K-1} \ell^p(G)$  such that*

$$\text{dist}_R(F) \leq 2 \left( 2 \int_2^{R/2} \left( \frac{t}{J(t)} \right)^p \frac{dt}{t} \right)^{1/p}.$$

*In particular, if  $J(t) = t/C$ , then*

$$\text{dist}_R(F) \leq 2C (2 \log(R/2))^{1/p}.$$

**Corollary 2.2.** *Assume that  $G_n$  has diameter  $\leq n$  and that  $J_{G,p}(t) \geq t/C$ , then,  $c_p(G_n) \leq 2C (2 \log(n/4))^{1/p}$ .*

On the other hand, we have proved in [T1] that the following finitely generated groups satisfy  $J_p(t) \geq t/C$  for some  $C < \infty$  and for all  $1 \leq p < \infty$ .

- the lamplighter group  $L_m = C_m \wr \mathbf{Z}$ ;

- solvable Baumslag-Solitar groups  $BS_m = \mathbf{Z}[1/m] \rtimes \mathbf{Z}$  for all  $m \in \mathbf{N}$ , where  $n \in \mathbf{Z}$  acts by multiplication by  $m^n$ ;
- polycyclic groups. Here, we will focus on the following example:  $SOL_A = \mathbf{Z}^2 \rtimes_A \mathbf{Z}$  where  $A$  is a matrix of  $SL_2(\mathbf{Z})$  with eigenvalues of modulus different from 1, e.g. the matrix  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ .

Note that respectively  $L_{m,n}$ ,  $BS_{m,n}$  and  $SOL_{A,n}$  are quotients of  $L_m$ ,  $BS_m$  and  $SOL_A$ .

### 3 Proofs of the main theorems

#### 3.1 Upper bounds

Thanks to Corollary 2.2, the upper bounds in Theorems 1, 2 and 3 follow from the following upper bounds on the diameters of the groups  $L_{m,n}$ ,  $BS_{m,n}$  and  $SOL_{A,n}$  (for the latter, see the discussion in Section 1.2).

**Proposition 3.1.** *We have*

- (i)  $\text{Diam}(L_{m,n}) \leq (m+3)n$ ;
- (ii)  $\text{Diam}(BS_{m,n}) \leq (m+1)n$ ;
- (iii) *Let  $N_n \simeq C_n^2$  be the kernel of  $SOL_{A,n} \rightarrow C_{o(A,n)}$ . Then, with the distance on  $N_n$  induced by the word distance on  $SOL_{A,n}$ , we have  $\text{Diam}(N_n) \leq c \log n$  for some  $c = c(A) > 0$ .*

**Proof:** For (i), see [Pa]. For (ii), note that every element of  $C_{m^n}$  can be written as

$$\sum_{i=0}^{n-1} a_i m^i = a_0 + m(a_1 + m(a_2 + \dots) \dots),$$

where  $0 \leq a_i \leq m-1$ . Finally, (iii) follows from the following well known lemma. ■

**Lemma 3.2.** *Let  $N \sim \mathbf{Z}^2$  be the kernel of  $SOL_A \rightarrow \mathbf{Z}$ . For all  $r \geq 1$ , denote by  $B_{N,SOL_A}(r)$  (resp.  $B_N(r)$ ), the ball of radius  $r$  for the metric on  $N$  induced by the word length on  $SOL_A$  (resp. for the usual metric on  $\mathbf{Z}^2$ ). There exists some  $\alpha = \alpha(A) < \infty$  such that*

$$B_N(1, e^{r/\alpha}) \subset B_{N,SOL_A}(r) \leq B_N(1, e^{\alpha r}).$$

**Proof:** Note that  $SOL_A$  embeds as a co-compact lattice in the connected solvable Lie group  $G = \mathbf{R}^2 \ltimes_A R$ , such that  $N$  maps on a (co-compact) lattice of  $\tilde{N} = \mathbf{R}^2$ . The lemma follows from the fact that  $\tilde{N}$  is the exponential radical of  $G$  (Guivarc'h [G] was the first one to introduce and to study the exponential radical of a connected solvable Lie group, without actually naming it, and this was rediscovered by Osin [O]). ■

## 3.2 Lower bounds

To obtain the lower bound on the distortion, we will need the following notion of relative girth.

**Definition 3.3.** Let  $\pi : G \rightarrow Q$  be a surjective homomorphism between two finitely generated groups and let  $S$  be a symmetric generating subset of  $G$ . Denote by  $X = (G, S)$  and  $Y = (H, \pi(S))$ . The relative girth  $g(Y, X)$  of  $Y$  with respect to  $X$  is the maximum integer  $n \in \mathbf{N}$  such that a ball of radius  $n$  in  $Y$  is isometric to a ball of radius  $n$  in  $X$ .

Recall [Bou] that the rooted binary tree  $T_n$  of dept  $n$  satisfies  $c_p(T_n) \geq c(\log n)^{1/p}$  for all  $2 \leq p < \infty$  and for some constant  $c > 0$ . The following remark follows trivially from this result and from the definition of relative girth.

**Proposition 3.4.** *We keep the notation of the previous definition. Assume that  $X$  contains a bi-Lipschitz embedded 3-regular tree. Then there exists some  $c > 0$  such that  $c_p(Y) \geq c(\log g(X, Y))^{1/p}$ .* ■

On the other hand, the groups  $L_m$ ,  $BS_m$  and  $SOL_A$  are solvable non-virtually nilpotent. Hence by [CT], they admit a bi-Lipschitz embedded 3-regular tree (for the lamplighter, see also [LPP]). So to prove the lower bounds of Theorems 1, 2 and 3, we just need to find convenient lower bounds for the relative girths, which is done by the following proposition.

**Proposition 3.5.** *We have*

- (i)  $g(L_{m,n}, L_m) \geq n$ ;
- (ii)  $g(BS_{m,n}, BS_m) \geq n$ ;
- (iii)  $g(SOL_{A,n}, SOL_A) \geq c \log n$  for some  $c = c(A) > 0$ .

**Proof:** The only non-trivial case, (iii), follows from Lemma 3.2. ■

## 4 Proof of Theorem 4

Let  $f_0$  be the dirac at 1, and for every integer  $1 \leq k \leq K$ , choose a function  $f_k \in \ell^p(G)$  such that

- the support of  $f_k$  is contained in the ball  $B(1, 2^k)$ ,
- $\|f_k\|_p \geq J(2^k)$
- $\sup_{s \in S} \|\lambda(s)f_k - f_k\|_p \leq 1$

For all  $v = (v_k)_{1 \leq k \leq n} \in K\ell^p(G)$  and all  $g \in G$ , define

$$\sigma(g)v = \bigoplus_k^{\ell^p} (\lambda(g)v_k + F_k)$$

where

$$F_k(g) = \left( \frac{2^k}{J(2^k)} \right) (f_k - \lambda(g)f_k).$$

Now consider the map  $F = \bigoplus^{\ell^p} b_k : G \rightarrow K\ell^p(G)$ . For all  $g \in G$ , we have

$$\begin{aligned} \|F(g)\|_p &= \|b(g)\|_p \\ &\leq \left( \sum_{k=0}^n \left( \frac{2^k}{J(2^k)} \right)^p \|\lambda(g)f_k - f_k\|_p^p \right)^{1/p} \\ &\leq \left( \sum_{k=0}^n \left( \frac{2^k}{J(2^k)} \right)^p \right)^{1/p} \\ &\leq |g|_S \left( \int_1^{\text{Diam}(G)/2} \left( \frac{t}{J(t/2)} \right)^p \frac{dt}{t} \right)^{1/p} \\ &= 2^{2/p} |g|_S \left( \int_1^{\text{Diam}(X)/4} \left( \frac{t}{J(t)} \right)^p \frac{dt}{t} \right)^{1/p}. \end{aligned}$$

On the other hand, since  $f_k$  is supported in  $B(1, 2^k)$ , if  $|g|_S \geq 2 \cdot 2^k$ , then the supports of  $f_k$  and  $\lambda(g)f_k$  are disjoint. Thus,

$$\begin{aligned} \|F(g)\|_p &= \|b(g)\|_p \\ &\geq \|b_k\|_p \\ &= 2^{1/p} \frac{2^k}{J(2^k)} \|f_k\|_p \\ &\geq 2^{1/p} 2^k, \end{aligned}$$

whenever  $d_S(x, y) \geq 2 \cdot 2^k$ . To conclude, we have to consider the case when  $g \in S \setminus \{1\}$ . But as  $f_0$  is a dirac at 1,  $\|F(g)\|_p \geq 1$ . So we are done. ■

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